Nonlinear Assimilation via Score-based Sequential Langevin Sampling

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Outline

Applications of Data Assimilation

Numerical weather prediction (NWP) uses **physical models** of the atmosphere and oceans to predict the weather based on the **measurement data**.

Physical models

Governing equations of atmosphere:

- conservation of momentum
- conservation of mass
- conservation of energy
- the equation of state for ideal gases
- conservation equation for water mass
- . . .

Measurement data

Problem Formulation

Dynamics model: A discrete time unobserved stochastic process $(\mathbf{X}_k)_{k\geq 1}$ satisfying

$$\mathbf{X}_{k+1} = \mathcal{F}_k(\mathbf{X}_k, \mathbf{V}_k) \qquad \rho_k(\mathbf{X}_{k+1}|\mathbf{X}_k) = p(\mathbf{X}_{k+1}|\mathbf{X}_k)$$

- $\mathcal{F}_k$ is a known time-dependent model.
- $\mathbf{V}_k$ is a random variable with known distribution.

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$$\mathbf{Y}_k = \mathcal{G}_k(\mathbf{X}_k, \mathbf{W}_k) \qquad g_k(\mathbf{Y}_k|\mathbf{X}_k) = p(\mathbf{Y}_k|\mathbf{X}_k)$$

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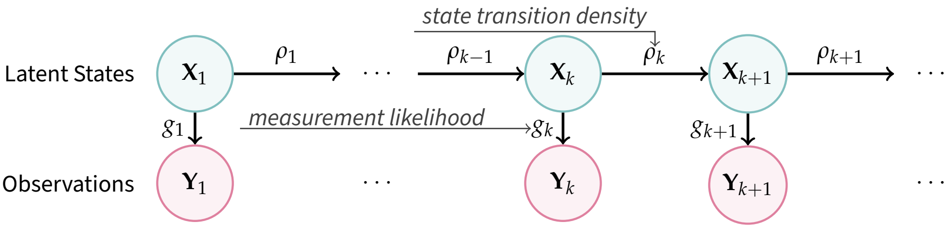

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Data Assimilation

The goal of the data assimilation is to estimate the posterior distribution of the latent state $\mathbf{X}_{k+1}$ given all available measurements $\mathbf{Y}_{[k+1]}$, that is, $\pi_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k+1]})$ for each $k \geq 1$.

Recursive Bayesian Filtering Framework

State-Space Model

Dynamics model	$\mathbf{X}_{k+1} = \mathcal{F}_k(\mathbf{X}_k, \mathbf{V}_k)$	$\longleftrightarrow$	State transition density	$\rho_k(\mathbf{X}_{k+1}\|\mathbf{X}_k)$
Measurement model	$\mathbf{Y}_{k+1} = \mathcal{G}_{k+1}(\mathbf{X}_{k+1}, \mathbf{W}_k)$	$\longleftrightarrow$	Measurement likelihood	$g_{k+1}(\mathbf{Y}_{k+1}\|\mathbf{X}_{k+1})$

$$
\begin{aligned}
&\pi_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k+1]}) \\
&\propto p(\mathbf{Y}_{k+1}|\mathbf{X}_{k+1}, \mathbf{Y}_{[k]})q_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k]}) \\
&= p(\mathbf{Y}_{k+1}|\mathbf{X}_{k+1}, \mathbf{Y}_{[k]}) \int p(\mathbf{X}_{k+1}|\mathbf{X}_k, \mathbf{Y}_{[k]})\pi_k(\mathbf{X}_k|\mathbf{Y}_{[k]})\mathrm{d}\mathbf{X}_k \\
&= g_{k+1}(\mathbf{Y}_{k+1}|\mathbf{X}_{k+1}) \int \rho_k(\mathbf{X}_{k+1}|\mathbf{X}_k)\pi_k(\mathbf{X}_k|\mathbf{Y}_{[k]})\mathrm{d}\mathbf{X}_k
\end{aligned}
$$

- Bayes' rule
- Chapman-Kolmogorov identity
- $\mathbf{Y}_{k+1} \perp\!\!\!\perp \mathbf{Y}_{[k]}|\mathbf{X}_{k+1}$, and $\mathbf{X}_{k+1} \perp\!\!\!\perp \mathbf{Y}_{[k]}|\mathbf{X}_k$

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$$\overbrace{\pi_{k+1}(\mathbf{X}_{k+1} \mid \mathbf{Y}_{[k+1]})}^{\text{current posterior}} \propto \underbrace{g_{k+1}(\mathbf{Y}_{k+1} \mid \mathbf{X}_{k+1})}_{\text{measurement likelihood}} \underbrace{\int \overbrace{\rho_k(\mathbf{X}_{k+1} \mid \mathbf{X}_k)}^{\text{state transition}} \overbrace{\pi_k(\mathbf{X}_k \mid \mathbf{Y}_{[k]})}^{\text{previous posterior}} \, \mathrm{d}\mathbf{X}_k}_{\text{current prior}}$$

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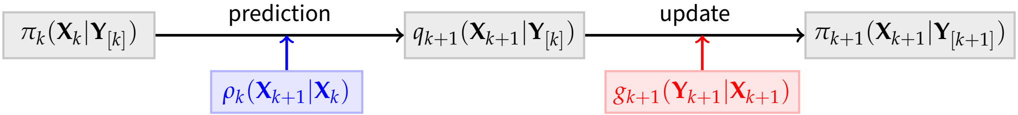
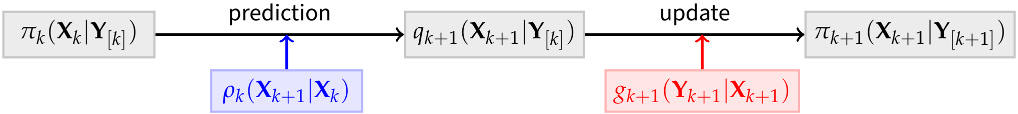


Classical Methods for Data Assimilation

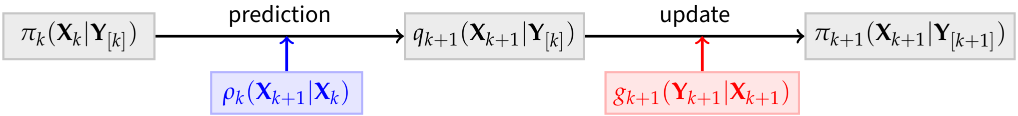
$$\underbrace{\pi_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k+1]})}_{\text{current posterior}} \propto g_{k+1}(\mathbf{Y}_{k+1}|\mathbf{X}_{k+1}) \int \rho_k(\mathbf{X}_{k+1}|\mathbf{X}_k) \underbrace{\pi_k(\mathbf{X}_k|\mathbf{Y}_{[k]})}_{\text{previous posterior}} \, \mathrm{d}\mathbf{X}_k$$



- How to estimate the prediction distribution $q_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k]})$?
- How to sample from the posterior distribution $\pi_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k+1]})$ given the prediction and likelihood?

Classical Methods for Data Assimilation

$$\underbrace{\pi_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k+1]})}_{\text{current posterior}} \propto g_{k+1}(\mathbf{Y}_{k+1}|\mathbf{X}_{k+1}) \int \rho_k(\mathbf{X}_{k+1}|\mathbf{X}_k) \underbrace{\pi_k(\mathbf{X}_k|\mathbf{Y}_{[k]})}_{\text{previous posterior}} \, \mathrm{d}\mathbf{X}_k$$



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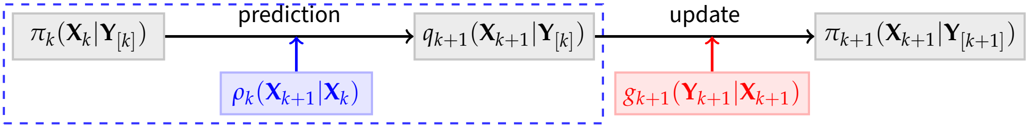
	$\pi_k(\mathbf{X}_k\|\mathbf{Y}_{[k]})$	$\rho_k(\mathbf{X}_{k+1}\|\mathbf{X}_k)$	$g_{k+1}(\mathbf{Y}_{k+1}\|\mathbf{X}_{k+1})$	Limitations
Kalman Filter	Gaussian approximation	Gaussian	Gaussian	linear + Gaussian
Particle Filter	particle approximation	–	–	particle degeneracy

Outline

Prediction and Score Matching

Prediction density $$q_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k]}) = \int \rho_k(\mathbf{X}_{k+1}|\mathbf{X}_k)\pi_k(\mathbf{X}_k|\mathbf{Y}_{[k]})\mathrm{d}\mathbf{X}_k$$



How to estimate the score function $\nabla \log q_{k+1}(\cdot|\mathbf{Y}_{[k]})$ *of the prediction distribution?*

$$q_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k]}) = \int \rho_k(\mathbf{X}_{k+1}|\mathbf{X}_k)\pi_k(\mathbf{X}_k|\mathbf{Y}_{[k]})\mathrm{d}\mathbf{X}_k$$

- Given: $\widehat{\mathbf{X}}_k^1, \dots, \widehat{\mathbf{X}}_k^n \sim^{\text{i.i.d.}} \widehat{\pi}_k(\mathbf{X}_k|\mathbf{Y}_{[k]}) \approx \pi_k(\mathbf{X}_k|\mathbf{Y}_{[k]})$.
- Prediction with **dynamics model**: $\underline{\mathbf{X}}_{k+1}^i = \mathcal{F}_k(\widehat{\mathbf{X}}_k^i, \mathbf{V}_k^i)$, for each $1 \leq i \leq n$.

$$\underline{\mathbf{X}}_{k+1}^i \sim \widehat{q}_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k]}) := \int \rho_k(\mathbf{X}_{k+1}|\mathbf{X}_k)\widehat{\pi}_k(\mathbf{X}_k|\mathbf{Y}_{[k]})\mathrm{d}\mathbf{X}_k \approx q_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k]}).$$

- Denoising **score matching**:

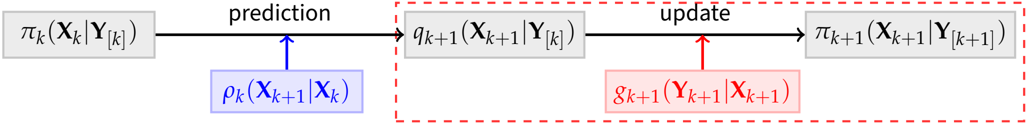
$$\widehat{\mathbf{s}}_{k+1}(\cdot|\mathbf{Y}_{[k]}) = \operatorname*{arg\,min}_{\mathbf{s}:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d} L_{k+1}(\mathbf{s}) = \mathbb{E}_{\underline{\mathbf{X}}_{k+1}\sim\widehat{q}_{k+1}(\cdot|\mathbf{Y}_{[k]})}\mathbb{E}_{\boldsymbol{\varepsilon}\sim N(0,\mathbf{I}_d)}\left[\|\sigma\mathbf{s}(\underline{\mathbf{X}}_{k+1}+\sigma\boldsymbol{\varepsilon},\mathbf{Y}_{[k]})+\boldsymbol{\varepsilon}\|_2^2\right]$$

which is the score function of the Gaussian smoothed prediction distribution

$$\widehat{\mathbf{s}}_{k+1}(\cdot|\mathbf{Y}_{[k]}) := \nabla\log\left(\int N(\cdot;\mathbf{X}_{k+1},\sigma^2\mathbf{I}_d)\widehat{q}_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k]})\mathrm{d}\mathbf{X}_{k+1}\right) \approx \nabla\log\widehat{q}_{k+1}(\cdot|\mathbf{Y}_{[k]})$$

Update via Langevin Sampling

Bayes' rule $$\pi_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k+1]}) \propto g_{k+1}(\mathbf{Y}_{k+1}|\mathbf{X}_{k+1}) q_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k]})$$



How to sample from the posterior distribution $\pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]})$ *given the prediction and likelihood?*

$$\pi_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k+1]}) \propto g_{k+1}(\mathbf{Y}_{k+1}|\mathbf{X}_{k+1})q_{k+1}(\mathbf{X}_{k+1}|\mathbf{Y}_{[k]})$$

- Langevin dynamics

$$\begin{aligned}\mathrm{d}\mathbf{Z}_t &= \nabla \log \pi_{k+1}(\mathbf{Z}_t|\mathbf{Y}_{[k+1]})\mathrm{d}t + \sqrt{2}\mathrm{d}\mathbf{B}_t, \\ &= \{\nabla \log g_{k+1}(\mathbf{Y}_{k+1}|\mathbf{Z}_t) + \nabla \log q_{k+1}(\mathbf{Z}_t|\mathbf{Y}_{[k]})\}\mathrm{d}t + \sqrt{2}\mathrm{d}\mathbf{B}_t, \quad \mathbf{Z}_0 \sim \widehat{q}_{k+1}(\cdot|\mathbf{Y}_{[k]}).\end{aligned}$$

  - Prediction score estimation
  - Euler-Maruyama discretization
- Score-based Langevin Monte Carlo

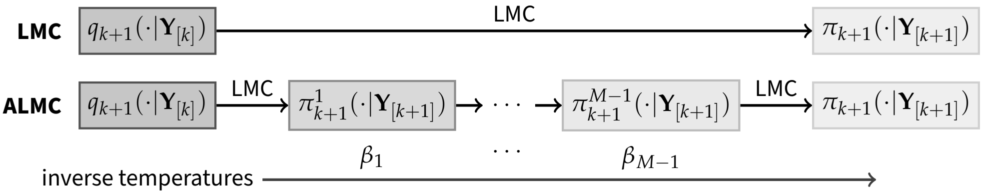
$$\widehat{\mathbf{Z}}_{(k+1)h} = \widehat{\mathbf{Z}}_{kh} + h\{\nabla \log g_{k+1}(\mathbf{Y}_{k+1}|\widehat{\mathbf{Z}}_{kh}) + \widehat{\mathbf{s}}_{k+1}(\widehat{\mathbf{Z}}_{kh}, \mathbf{Y}_{[k]})\} + \sqrt{2h}\boldsymbol{\xi}_k, \quad \widehat{\mathbf{Z}}_0 \sim \widehat{q}_{k+1}(\cdot|\mathbf{Y}_{[k]}).$$

Annealing Strategy

Interpolations between the prediction and posterior distribution:

$$\pi_{k+1}^{m}(\cdot|\mathbf{Y}_{[k+1]}) \propto \pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]})^{\beta_m} q_{k+1}(\cdot|\mathbf{Y}_{[k]})^{1-\beta_m}, \quad 0 \leq m \leq M$$

- Annealing schedule: $0 \equiv \beta_0 < \beta_1 < \cdots < \beta_M \equiv 1$.



Interpolations between the prediction and posterior distribution:

$$\pi_{k+1}^{m}(\cdot|\mathbf{Y}_{[k+1]}) \propto \pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]})^{\beta_m} q_{k+1}(\cdot|\mathbf{Y}_{[k]})^{1-\beta_m}, \quad 0 \le m \le M$$

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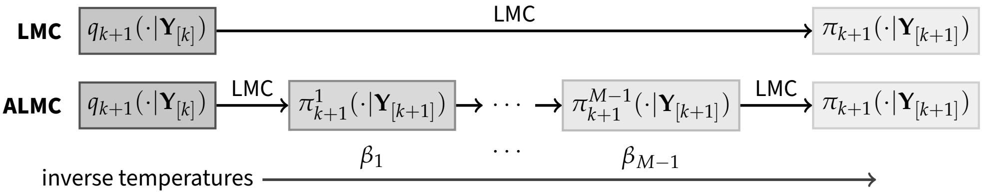
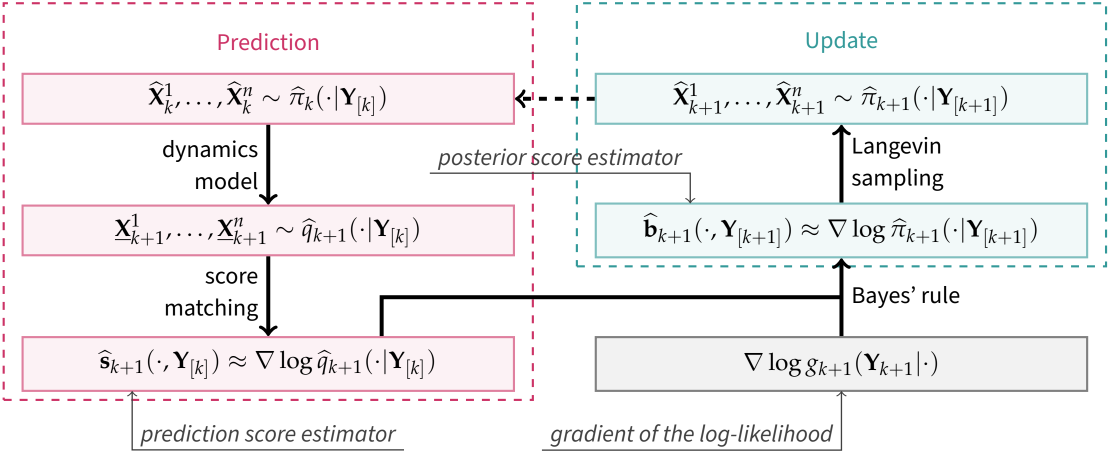


$$\begin{aligned}\nabla \log \pi_{k+1}^{m}(\cdot|\mathbf{Y}_{[k+1]}) &= \beta_m \nabla \log \pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]}) + (1-\beta_m)\nabla \log q_{k+1}(\cdot|\mathbf{Y}_{[k]}) \\ &= \beta_m \nabla \log g_{k+1}(\mathbf{Y}_{k+1}|\cdot) + \nabla \log q_{k+1}(\cdot|\mathbf{Y}_{[k]})\end{aligned}$$

Summary of Procedure



Algorithm 1: Score-based **S**equential **L**angevin **S**ampling for data assimilation.

Input: The observations $(\mathbf{Y}_k)_{k\in\mathbb{N}}$, the dynamics model $(\mathcal{F}_k)_{k\in\mathbb{N}}$, the likelihood $\{g_k(\mathbf{Y}_k|\cdot)\}_{k\in\mathbb{N}}$.

Output: A particle approximation $\widehat{\mathbf{X}}_{k+1}^1, \ldots, \widehat{\mathbf{X}}_{k+1}^n$ to the distribution $\pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]})$.

```
# Initial posterior sampling.
```
Sample from the gauss of the initial prior distribution $\underline{\mathbf{X}}_1^1, \ldots, \underline{\mathbf{X}}_1^n \sim^{\text{i.i.d.}} \widehat{q}_1$.

Estimate the score from $\{\underline{\mathbf{X}}_1^i\}_{i=1}^n$ by score matching $\widehat{\mathbf{s}}_1$.

Sample from the posterior $\widehat{\pi}_1(\cdot|\mathbf{y}_1)$ by Langevin sampling: $\widehat{\mathbf{X}}_1^1, \ldots, \widehat{\mathbf{X}}_1^n \leftarrow \mathrm{ALMC}(\underline{\mathbf{X}}_1^1, \ldots, \underline{\mathbf{X}}_1^n, \widehat{\mathbf{s}}_1, g_1(\mathbf{Y}_1|\cdot))$.

```
# Recursive posterior sampling.
```
for $\underline{k = 1, 2, \ldots}$ **do**

```
# Prediction step.
```
Run the dynamics model: $\underline{\mathbf{X}}_{k+1}^i \leftarrow \mathcal{F}_k(\widehat{\mathbf{X}}_k^i, \mathbf{V}_k^i)$ with $\mathbf{V}_k^i \sim p_{\mathbf{V}}$ for $1 \le i \le n$.

Estimate the prediction score from $\{\underline{\mathbf{X}}_{k+1}^i\}_{i=1}^n$ by score matching $\widehat{\mathbf{s}}_{k+1}(\cdot, \mathbf{Y}_{[k]})$.

```
# Update step.
```
Sample from the posterior $\pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]})$ by Langevin sampling:

$\widehat{\mathbf{X}}_{k+1}^1, \ldots, \widehat{\mathbf{X}}_{k+1}^n \leftarrow \mathrm{ALMC}(\underline{\mathbf{X}}_{k+1}^1, \ldots, \underline{\mathbf{X}}_{k+1}^n, \widehat{\mathbf{s}}_{k+1}(\cdot, \mathbf{Y}_{[k]}), g_{k+1}(\mathbf{Y}_{k+1}|\cdot))$.

end

return $\underline{\widehat{\mathbf{X}}_{k+1}^1, \ldots, \widehat{\mathbf{X}}_{k+1}^n}$

Outline

Assumptions

- **A1. Lipschitz score.** For each $k \in \mathbb{N}$, the posterior score is λ-Lipschitz on $\mathbb{R}^d$, that is, for each $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$,

$$\|\nabla \log \pi_{k+1}(\mathbf{x}_1|\mathbf{Y}_{[k+1]}) - \nabla_{\mathbf{x}} \log \pi_{k+1}(\mathbf{x}_2|\mathbf{Y}_{[k+1]})\|_2 \leq \lambda \|\mathbf{x}_1 - \mathbf{x}_2\|_2.$$

- **A2. Log-Sobolev inequality.** For each $k \in \mathbb{N}$, the posterior distribution satisfies a log-Sobolev inequality with constant C_{LSI}, that is, for each function $f \in C_0^\infty(\mathbb{R}^d)$,

$$\mathrm{Ent}(f^2) \leq 2C_{\mathrm{LSI}} \mathbb{E}\left[\|\nabla f\|_2^2\right],$$

  where the entropy is defined as $\mathrm{Ent}(g) = \mathbb{E}[g \log g] - \mathbb{E}[g] \log \mathbb{E}[g]$, and the expectation is taken with respect to the posterior distribution $\pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]})$.

- **A3. Boundedness and condition number.** For each $k \in \mathbb{N}$,
  - there exists $B \geq 1$ such that $\rho_k(\mathbf{x}|\mathbf{X}_k), \|\nabla \rho_k(\mathbf{x}|\mathbf{X}_k)\|_\infty \leq B$ for $\mathbf{x} \in \mathbb{R}^d$,
  - there exists $D \geq 1$ such that $q_{k+1}(\mathbf{x}|\mathbf{Y}_{[k]}) \geq D^{-1}$ for $\mathbf{x} \in \Omega_{k+1} := \mathrm{supp}(q_{k+1}(\cdot|\mathbf{Y}_{[k]}))$,
  - $q_1(\mathbf{x}), \|\nabla q_1(\mathbf{x})\|_\infty \leq B$ and $q_1(\mathbf{x}) \geq D^{-1}$ for $\mathbf{x} \in \Omega_1 := \mathrm{supp}(q_1)$, and
  - there exists $\kappa > 0$ such that

$$\frac{\sup_{\mathbf{x}} g_1(\mathbf{Y}_1|\mathbf{x})}{\int g_1(\mathbf{Y}_1|\mathbf{x}) q_1(\mathbf{x}) \mathrm{d}\mathbf{x}}, \frac{\sup_{\mathbf{x}} g_{k+1}(\mathbf{Y}_{k+1}|\mathbf{x})}{\int g_{k+1}(\mathbf{Y}_{k+1}|\mathbf{x}) q_{k+1}(\mathbf{x}|\mathbf{Y}_{[k]}) \mathrm{d}\mathbf{x}} \leq \kappa.$$

- **A4. Error of score matching.** There exists a score matching tolerance $\Delta \in (0, 1)$ such that

$$\mathbb{E}_{\underline{\mathbf{X}}_1}\left[\|\nabla_{\mathbf{x}} \log \widehat{q}_1(\underline{\mathbf{X}}_1) - \widehat{\mathbf{s}}_1(\underline{\mathbf{X}}_1)\|_2^2\right] \leq \Delta^2,$$

$$\mathbb{E}_{\underline{\mathbf{X}}_{k+1}}\left[\|\nabla_{\mathbf{x}} \log \widehat{q}_{k+1}(\underline{\mathbf{X}}_{k+1}|\mathbf{Y}_{[k]}) - \widehat{\mathbf{s}}_{k+1}(\underline{\mathbf{X}}_{k+1}, \mathbf{Y}_{[k]})\|_2^2\right] \leq \Delta^2,$$

for each $k \in \mathbb{N}$. Here the expectation $\mathbb{E}_{\underline{\mathbf{X}}_1}[\cdot]$ is taken with respect to $\underline{\mathbf{X}}_1 \sim \widehat{q}_1$, and the expectation $\mathbb{E}_{\underline{\mathbf{X}}_{k+1}}[\cdot]$ is taken with respect to $\underline{\mathbf{X}}_{k+1} \sim \widehat{q}_{k+1}(\cdot|\mathbf{Y}_{[k]})$.

Convergence Analysis for Posterior Sampling

Under Assumptions **A1** to **A4**. Then for each $k \in \mathbb{N}$ and each terminal time $T = Kh$,

$$\underbrace{\|\pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]}) - \widehat{\pi}_{k+1}(\cdot|\mathbf{Y}_{[k+1]})\|_{\mathrm{TV}}^2}_{\text{posterior error}} \lesssim \underbrace{\exp\left(-\frac{T}{5C_{\mathrm{LSI}}}\right)\eta_\chi^2}_{\text{convergence of LD}} + \underbrace{dC_{\mathrm{LSI}}\lambda^2 h}_{\text{discretization error}} + \underbrace{\sqrt{\kappa B^3 D^3}(T + C_{\mathrm{LSI}}\eta_\chi)\Delta}_{\text{score estimation error}}$$
$$+ B^4 D^4 T \underbrace{\|\pi_k(\cdot|\mathbf{Y}_{[k]}) - \widehat{\pi}_k(\cdot|\mathbf{Y}_{[k]})\|_{\mathrm{TV}}^2}_{\text{prior error}},$$

where the step size h and the initial distribution $\pi_k^0(\cdot|\mathbf{Y}_{[k]})$ satisfies

$$h \lesssim \frac{1}{dC_{\mathrm{LSI}}\lambda^2}, \quad \chi^2(\pi_{k+1}^0(\cdot|\mathbf{Y}_{[k+1]})\|\pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]})) \leq \eta_\chi^2.$$

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where the step size h and the initial distribution $\pi_k^0(\cdot|\mathbf{Y}_{[k]})$ satisfies

$$h \lesssim \frac{1}{dC_{\mathrm{LSI}}\lambda^2}, \quad \chi^2(\pi_{k+1}^0(\cdot|\mathbf{Y}_{[k+1]})\|\pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]})) \leq \eta_\chi^2.$$

- Early-stopping: Trade-off between the convergence of Langevin dynamics and score estimation error.
- Condition number: How the score matching error Δ effects the posterior error.

Convergence Analysis for Assimilation

Under Assumptions **A1** to **A4**. Suppose that the initial prior error satisfy

$$\varepsilon_{\text{init}} := \|\nabla_{\mathbf{x}} \log q_1 - \nabla_{\mathbf{x}} \log \widehat{q}_1\|_{L^\infty(\mathbb{R}^d)}.$$

Then for each time step $k \in \mathbb{N}$ and error tolerance $\varepsilon \in (0,1)$,

$$\|\pi_{k+1}(\cdot|\mathbf{Y}_{[k+1]}) - \widehat{\pi}_{k+1}(\cdot|\mathbf{Y}_{[k+1]})\|_{\text{TV}} \leq \widetilde{\mathcal{O}}(\varepsilon_{\text{init}} + \varepsilon),$$

where the $\widetilde{\mathcal{O}}$ notation omits logarithmic factors of $\varepsilon_{\text{init}}$ and ε, and the constant behind the $\widetilde{\mathcal{O}}$ notation is independent of $\varepsilon_{\text{init}}$ and ε. Moreover, the step size h, the number of the Langevin iterations K and the score matching error Δ satisfy:

$$h = \Theta\Big(\frac{\varepsilon^2}{dC_{\text{LSI}}\lambda^2}\Big), \quad K = \Theta\Big(\frac{d\lambda^2 C_{\text{LSI}}^2}{\varepsilon^2} \log\Big(\frac{\eta_\chi^2}{\varepsilon^2}\Big)\Big),$$

$$\Delta = \Theta\Big(\frac{\varepsilon^2}{\sqrt{\kappa B^3 D^3} C_{\text{LSI}}} \frac{1}{\log(\varepsilon^{-2}\eta_\chi^2) + \eta_\chi}\Big).$$

Outline

Double-Well Potential

Dynamics model:

$$dX_t = -\nabla U(X_t)dt + \beta dB_t$$

- Double-well potential: $U(x) = x^4 - 2x^2$.

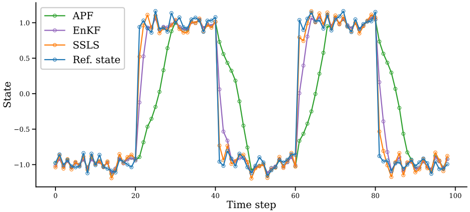
Measurement model:

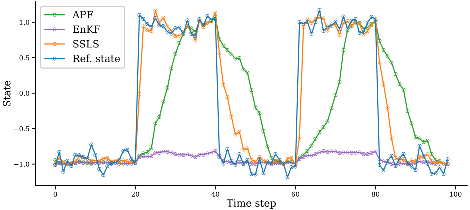
- Linear measurement model:

$$Y_k = X_k + \sigma W_k$$

- Nonlinear measurement model:

$$Y_k = \exp(X_k - \gamma_k) + \sigma W_k$$





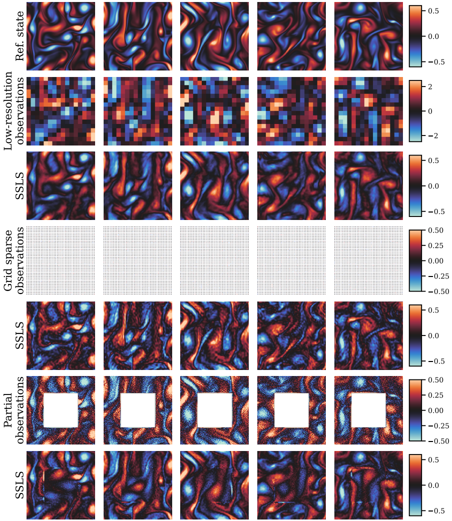
Kolmogorov flow

Dynamics model:

$$\begin{cases} \partial_t \mathbf{u} = -(\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} - \frac{1}{\rho} \nabla p + \mathbf{F}, \\ 0 = \nabla \cdot \mathbf{u}. \end{cases}$$

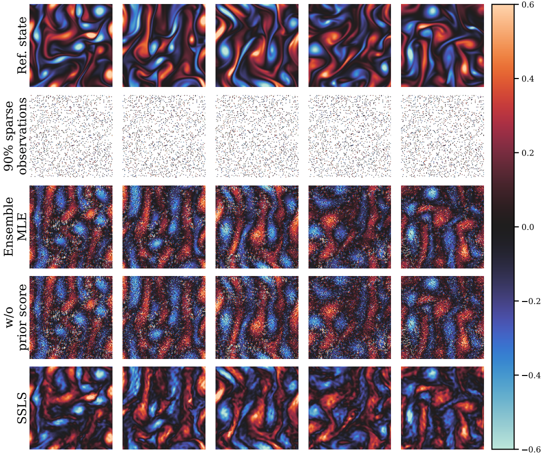
Measurement model:

- Low-resolution observation
- Sparse observation
- Partial observation

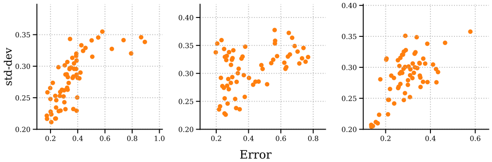


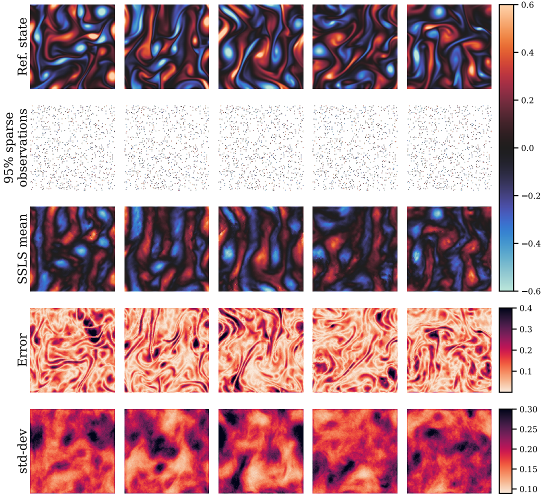
Ablation study: influence of the prediction score

- Score-based Langevin sampling
- Langevin sampling without prediction score
- Ensemble maximum likelihood estimation



Uncertainty quantification





Outline

Concluding Remarks

- Improvement of the computational efficiency.
- Provable non-log-concave posterior sampling.
- Refined error bounds for long-time assimilation.
- Applications in the real data.

Reference: Zhao Ding, Chenguang Duan, Yuling Jiao, Jerry Zhijian Yang, Cheng Yuan, and Pingwen Zhang. Nonlinear Assimilation via Score-based Sequential Langevin Sampling. arXiv:2411.13443.

Thanks for your attention!

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